Thermodynamics of multi-boson phenomena

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Using the method of locally equilibrium statistical operator we consider the thermalized relativistic quantum fields in an oscillatory trap. We compare this thermal picture of the confined boson gas with non-relativistic model of independent factorized sources. We find that they are equivalent in the limit of very large effective sizes R, more exactly, when the Compton wave length $1/m$ and thermal wave length $1/\sqrt{mT}$ are much smaller than R . Under this conditions we study the influence of Bose condensation in finite volumes on the structure of the Wigner function, momentum spectra and correlation function.

I. INTRODUCTION

In heavy ion collisions at RHIC and LHC energies quasi-macroscopical systems containing $10^4 - 10^5$ particles are expected to be created. If phase-space densities of such systems at a pre-decaying stage will be high enough, one can observe multi-boson effects enhancing the production of pions with low relative momenta, softening their spectra and modifying correlation functions. One can even hope to observe new interesting phenomena like boson condensation in certain kinematic regions with a large pion density in the 6-dimensional phase space: $f = (2\pi)^3 d^6n/d^3\mathbf{p}d^3\mathbf{x} >\sim 1$ (see, e.g., [1–5]).

Generally, the account of the multi-boson effects is extremely difficult task. Even on the neglection of particle interactions in the final state the requirement of the BE symmetrization leads to severe numerical problems which increase factorially with the number of produced bosons [1,2]. In such situation, it is important that there exists a simple analytically solvable models [3] allowing for a study of the characteristic features of the multi-boson systems. Actually, there are two basic methods that are presently used to describe multi-boson systems in finite (small) volumes typical for A+A collisions. First one [3,6,7] is maximally closed to the procedure of computer simulations of multi-boson effects. We call it: the model of independent factorized sources, or MIFS. It implies the independent emission of 'probed' non-identical particles with factorized Wigner functions

$$D_n(p_1, x_{1;}...; p_n, x_n) = \prod_{i=1}^{n} D(p_i, x_i). \tag{1}$$

The basic function in such an approach is the 'probed' Wigner distribution $D(p, x)$ that is chosen usually in the thermal-like (Boltzmann) form:

$$D(p, x) = \frac{\eta}{(2\pi R\Delta)^3} \exp(-\frac{\mathbf{p}^2}{2\Delta^2} - \frac{\mathbf{r}^2}{2R^2})\delta(t). \tag{2}$$

We normalize it to a mean multiplicity η. In typical cases the input numbers n of particles are described by the Poissonian distribution

$$P(n) = e^{-\eta}\eta^n/n!. \tag{3}$$

In order to obtain an output ('true') particle number distribution as well as single and two-particle momentum spectra, the special procedure of "switching on" of the Bose-statistics is applied. In other words, the symmetrization of the 'probe' n-particle normalized amplitudes $A_n\{p_i\}$ is provided for. Then the final multiplicity distribution is $P^S(n) = P(n)N[A^S_n]$, where $N[A^S_n]$ is the normalized weight due to the symmetrization.

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The other approach [8,9]deals with true boson statistics from very beginning and is based on the density matrix formalism. In particular, the specific $\rho-$matrix ansatz in the wave packet formalism was proposed [9] to get (and develop) MIFS results.

To treat the results of these approaches one can often use the statistical thermodynamics language like "rare gas", "Bose condensate", etc. At the same time no systematic consideration based on the thermal matrix density for such a kind of system in finite volumes has been done. For thermal relativistic essentially finite (small) systems the problem is mathematically rather complicated, however for large enough ones it can be solved analytically. We will demonstrate it here using the method of statistical operator.

II. DENSITY MATRIX FOR LOCALLY EQUILIBRIUM SYSTEMS

According to the definition the statistical operator is

$$\rho = e^{-S}. \tag{4}$$

For locally equilibrium systems the entropy S has to be maximized under additional conditions fixing the average densities of energy $\epsilon(x)$, momentum $\mathbf{p}(x)$, charge $q(x)$. Systems are considered on some hypersurface $\sigma : d\sigma_\nu = d\sigma n_\nu$ with a time-like normal vector n^ν. In the relativistic covariant form the conditions look like

$$p^\mu(x) \equiv (\epsilon(x), \mathbf{p}(x)) = \left\langle n_\nu(x)\widehat{T}^{\mu\nu}(x)\right\rangle, \quad q(x) = \left\langle n_\nu(x)\widehat{J}^\nu(x)\right\rangle \tag{5}$$

where $\langle ... \rangle$ means the average with the statistical operator ρ in (4). The result for entropy operator is then [10], [11]

$$S = S(\sigma) = \Phi(\sigma) + \int d\sigma_\nu(x)\beta_\mu(x)\widehat{T}^{\mu\nu}(x) - \int d\sigma_\nu(x)\mu(x)\widehat{J}^\nu(x), \tag{6}$$

where $\Phi(\sigma) = \ln\ Sp\exp\{\int d\sigma\ n_\nu(x)\beta_\mu(x)\widehat{T}^{\mu\nu}(x)\}$ is Masier-Planck functional, $\beta_\mu(x)$ and $\mu(x)$ are Lagrange multipliers. For real one-component free scalar field we will use the current of particle number density $\widehat{J}^\nu(x) = \varphi^{(+)}(x)\ \overleftrightarrow{\partial^\nu}\ \varphi^{(-)}(x)$ where $\varphi^{(+)}$ and $\varphi^{(-)}$ are the positive and negative field components. The energy-momentum tensor has the standard form.

Then for the system which has no internal flows and is in local equilibrium state on the hypersurface $t = 0$: $n^\nu = (1, \mathbf{0})$, at the same temperature $T : \beta_\mu(x) = (\frac{1}{T}, \mathbf{0})$ the density matrix looks like:

$$\rho = \frac{1}{Z}\exp\left[-\beta\int d^3pp^0a_p^\dagger a_p + \frac{\beta}{2(2\pi)^3}\int d^3x\mu(x)\frac{d^3kd^3k'}{k_0k_0'}(k_0+k_0')e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}}a_k^\dagger a_{k'}\right] \tag{7}$$

To guarantee the maximal closeness to the MIFS we will use the chemical potential in the form:

$$\mu(x) = \mu - \frac{\mathbf{x^2}}{2R^2\beta}. \tag{8}$$

The description of inclusive spectra and correlations for a multiparticle production is based on a computation of the averages

$$p^0\frac{dN}{d\mathbf{p}} = \left\langle a_p^+a_p\right\rangle,\ p_1^0p_2^0\frac{dN}{d\mathbf{p_1}d\mathbf{p_2}} = \left\langle a_{p_1}^+a_{p_2}^+a_{p_1}a_{p_2}\right\rangle,\ \text{etc,...} \tag{9}$$

For Gaussian-type operator like (7) the thermal Wick theorem takes place:

$$\left\langle a_{p_1}^+a_{p_2}^+a_{p_1}a_{p_2}\right\rangle = \left\langle a_{p_1}^+a_{p_1}\right\rangle\left\langle a_{p_2}^+a_{p_2}\right\rangle + \left\langle a_{p_1}^+a_{p_2}\right\rangle\left\langle a_{p_2}^+a_{p_1}\right\rangle. \tag{10}$$

To find the averages $\left\langle a_{p_1}^+a_{p_2}\right\rangle$ the Gaudin's method [12] is used [13]. As a result we have:

$$\left\langle a_{p_1}^+a_{p_2}\right\rangle = p_2^0\sum_{n=1}^{\infty}G_n(p_1,p_2);\ G_n(p_1,p_2) = \int d^3kG_{n-1}(p_1,k)G_1(k,p_2). \tag{11}$$

The basic function $G_1(p_1, p_2)$ will be defined below. For this aim, first the commutation relation with entropy operator have to be considered:

$$[a(p),S]=\int d^3kM(p,k)a(k)\Longrightarrow M(p,k)\approx M^{(0)}(p,k)+O(\frac{1}{p_0^2R^2})+O(\frac{\beta}{p_0R^2}) \tag{12}$$

where, using (7) and (8), we have

$$M^{(0)}(p_1,p_2)=\beta p_2^0\delta(\mathbf{p}_1-\mathbf{p}_2)-\frac{\beta}{(2\pi)^3}\int d^3xe^{-i(\mathbf{p}_1-\mathbf{p}_2)\mathbf{x}}\mu(x). \tag{13}$$

The basic function $G_1(p_1,p_2)$ is then defined as follows:

$$G_1(p_1,p_2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}M_n^*(p_1,p_2), \tag{14}$$

where

$$\begin{aligned} &M_0(p_{1,}\,p_2)=\delta(\mathbf{p}_1-\mathbf{p}_2),\;\; M_1(p_{1,}\,p_2)=M(p_{1,}\,p_2);\\ &M_n(p_1,p_2)=\int d^3kM_{n-1}(p_1,k)M_1(k,p_2). \end{aligned} \tag{15}$$

III. THE COMPARISON WITH MIFS.

At $m^2R^2, m\beta^{-1}R^2\gg 1$ we find, neglecting the terms $1/p_0^4R^4$ and $\beta^2/p_0^2R^4$ in Eqs.(12),

$$\begin{aligned} &G_1(p_1,p_2)=G_1^{(0)}(p_1,p_2)+G_1^{(1)}(p_1,p_2)+O(\tfrac{1}{p_0^4R^4},\tfrac{\beta^2}{p_0^2R^4})\approx\\ &\tfrac{1}{(2\pi)^3}\int d^3xe^{i\mathbf{qx}}\exp(-\beta(p_1^0-\mu(x)))\times\\ &\left[1-\tfrac{3\beta}{4p_1^0R^2}(1+\tfrac{2i\mathbf{p}_1\mathbf{x}}{3}+\tfrac{2\beta\mathbf{p}_1^2}{9p_1^0})-\tfrac{3}{4(p_1^0R)^2}(1+\tfrac{\beta\mathbf{p}_1^2}{p_1^0})\right]=\int d^3xe^{i\mathbf{qx}}D(x,p). \end{aligned} \tag{16}$$

Here $p=(p_1+p_2)/2$; $q=p_1-p_2$. At $m^2R^2,\Delta^2R^2\to\infty$ in the non-relativistic limit the Wigner distribution is

$$D(x,p)\approx\frac{\xi}{(2\pi)^3}\exp(-\frac{\mathbf{p}^2}{2\Delta^2}-\frac{\mathbf{r}^2}{2R^2}), \tag{17}$$

where $\Delta^2=mT$ and fugacity $\xi=e^{\widetilde{\mu}},\widetilde{\mu}=\mu-m$. Comparing Eqs. (17) and (2) we found that in the non-relativistic approach in the limit of very large emission volumes the thermal density matrix (7) with the chemical potential (8) and the fugacity $\xi=\eta/(\Delta R\ )^3$ is equivalent to MIFS. The physical reasons are the following. First, if the wave-length of the quanta is much less that the system size, the particle does not 'feel' the finiteness of the systems and the 'probe' distribution will be the same as in the thermodynamic limit, i.e., the Boltzmann-like one. Second, for large systems the assumption of an independent emission of non-interacting particles is natural: the average distance between any two particles is large comparing with its wave-length (or with the size of the wave packet).

At the same time Eq.(16) indicates that for essentially small system's sizes compared with quanta wave-lengths, the single-particle locally equilibrium distribution $D(x,p)$ has no more simple Boltzmann-like form (2). Some distortion terms have relativistic nature and are essential when the system size is close to the Compton wave-length, $R\sim 1/m$. The others can take place even if $Rm\gg 1$, depending on the ratio between size R and thermal wave-length $1/\sqrt{mT}$. They are of quantum nature and appear when one describes the thermal equilibrium of the quanta with an average de Broglie wave length that is larger than the system size. In general, according to Eq.(16) one can expect the reduction of soft quanta in thermal model in comparison with the MIFS Boltzmann anzats while the distributions of "hard" quanta coincide in both approaches.

IV. THE WIGNER FUNCTIONS AND SPECTRA IN MULTI-BOSON SYSTEMS

We will consider the limiting behavior of the functions $G_n(\mathbf{p}_1,\mathbf{p}_2)$ (11) at small $n << \Delta R$ and large $n \gg \Delta R$ at $R\Delta \gg 1$ in the non-relativistic case. Neglecting the terms $1/\Delta^2 R^2$ and $1/m^2R^2$ in Eq. (16) we can put $G_1(p_1,p_2) = G_1^{(0)}(p_1,p_2)$ and found the optimal tailing of the two limiting forms at the point $n_t = R\Delta$. Then at $R\Delta \gg 1$ one can express the basic operator average (11) through the Wigner functions of the Bose gas (g) and the Bose condensate (c) (we will see the correspondence later) as follows:

$$\frac{1}{p_2^0}\left\langle a_{p_1}^+ a_{p_2}\right\rangle = \sum_{n=1}^{\infty} G_n(p_1,p_2) \approx \int d^3x e^{i\mathbf{qx}}(f_g(p,x)+f_c(p,x)), \tag{18}$$

where

$$f_g(p,x) = \frac{1}{(2\pi)^3}\frac{1}{\widetilde{\xi}_x^{-1}\exp(\mathbf{p}^2/2\Delta^2)-1}\ ,$$
$$\widetilde{\xi}_x = \widetilde{\xi}\exp(-\mathbf{x}^2/2R^2) \equiv e^{\beta(\widetilde{\mu}-\frac{3}{2\Delta R})}\exp(-\mathbf{x}^2/2R^2) \tag{19}$$

and

$$f_c(p,x) = \frac{\widetilde{\xi}}{1-\widetilde{\xi}}\widetilde{\xi}^{R\Delta}\frac{1}{\pi^3}\exp(-\frac{R}{\Delta}\mathbf{p}^2 - \frac{\Delta}{R}\mathbf{x}^2). \tag{20}$$

We have used here the following transformation:

$$\exp\left(-\frac{\mathbf{q}^2R^2}{2n}\right) = \left(\frac{n}{2\pi R^2}\right)^{3/2}\int d^3x e^{-i\mathbf{xq}}\exp(-\frac{\mathbf{x}^2 n}{2R^2}). \tag{21}$$

Note here that the critical value of the nonrelativistic chemical potential $\widetilde{\mu} = \frac{3}{2\Delta R}$ is positive because gas is not ideal (effectively the gas interacts with external field confining it in a finite volume).

In the thermodynamic limit: volume $V \to \infty$ ($R\Delta \to \infty$) at fixed temperature T and at fixed particle density $n = N/V$ the phase space density in the central part is

$$\widetilde{n} \equiv \frac{n(0)}{\Delta^3} \approx \frac{1}{(2\pi)^{3/2}}\Phi_{3/2}(\widetilde{\xi}) + \frac{1}{(\pi R\Delta)^{3/2}}\frac{\widetilde{\xi}}{1-\widetilde{\xi}}\widetilde{\xi}^{R\Delta} \equiv \widetilde{n}_g + \widetilde{n}_c \tag{22}$$

where $\Phi_\alpha(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^\alpha}$. The first term corresponds to the ordinary Bose-Einstein gas contribution. Let us introduce associated with it at $\widetilde{\xi} = 1$ the critical density: $\widetilde{n}_{cr} = \frac{1}{(2\pi)^{3/2}}\Phi_{3/2}(1)$. It is easy to see that in order to move to the thermodynamic limit at fixed density $\widetilde{n} > \widetilde{n}_{cr}$ the parameter $\widetilde{\xi}$ at large enough $R\Delta$ has to tend to unity as:

$$\widetilde{\xi} = 1 - \frac{1}{(\widetilde{n}-\widetilde{n}_{cr})(\pi R\Delta)^{3/2}}. \tag{23}$$

The single particle inclusive distribution in the thermodynamic limit has the form:

$$n^{(1)}(\mathbf{p}) = \int_{-\infty}^{\infty} d^3x f(\mathbf{x},\mathbf{p}) = n_g(\mathbf{p}) + n_c(\mathbf{p}) \approx$$
$$\frac{R^3}{(2\pi)^{3/2}}\Phi_{3/2}(\widetilde{\xi}e^{-\frac{\mathbf{p}2}{2\Delta^2}}) + \frac{\widetilde{\xi}}{1-\widetilde{\xi}}\widetilde{\xi}^{R\Delta}R^3(\pi R\Delta)^{-3/2}\exp(-\frac{R}{\Delta}\mathbf{p}^2) \tag{24}$$
$$\xrightarrow{R\Delta\to\infty} \frac{1}{(2\pi)^3}\int d^3x\frac{1}{\widetilde{\xi}_x^{-1}\exp(\mathbf{p}^2/2\Delta^2)-1} + (\pi R\Delta)^{3/2}(\widetilde{n}-\widetilde{n}_{cr})\delta(\mathbf{p})$$

Two terms in the last equality of Eq. (24) are associated with the classical Bose gas and the condensate in thermodynamic limit at the densities $\widetilde{n} > \widetilde{n}_{cr}$. They describe Bose gas and Bose condensate for finite systems at the condition $R\Delta >> 1$. Note that for Bose condensate the momentum distribution is much more narrow than for Bose gas, $\sqrt{\frac{\Delta}{2R}} \ll \Delta$.

V. THE INTERFEROMETRY OF MULTI-BOSON SOURCES

The calculation of two-particle inclusive spectra is based on Eqs.(9), (10), (18). One can easily find that

$$\frac{1}{p_2^0}\left\langle a_{p_1}^+ a_{p_2}\right\rangle = n_g(\mathbf{p})\exp(-\mathbf{q}^2 R_g^2/2) + n_c(\mathbf{p})\exp(-\mathbf{q}^2 R_c^2/2), \tag{25}$$

where

$$R_g^2 = R^2\Phi_{5/2}(\widetilde{\xi} e^{-\frac{p2}{2\Delta^2}})/\Phi_{3/2}(\widetilde{\xi} e^{-\frac{p2}{2\Delta^2}}) \overset{p=0,\widetilde{\xi}=1}{\longrightarrow} \approx R^2/2,\ R_c^2 = R/2\Delta. \tag{26}$$

The correlation function at small q^2 limit, $qR \ll 1$, is then:

$$C(p,q) = 1 + \frac{\left\langle a_{p_1}^+ a_{p_2}\right\rangle \left\langle a_{p_2}^+ a_{p_1}\right\rangle}{\left\langle a_{p_1}^+ a_{p_1}\right\rangle \left\langle a_{p_2}^+ a_{p_2}\right\rangle} = 1 + \exp\left(-\frac{n_g(\mathbf{p})}{n_g(\mathbf{p}) + n_c(\mathbf{p})} R_g^2 \mathbf{q}^2\right). \tag{27}$$

At large q^2 limit, $qR \gg 1$, it is

$$C(p,q) = 1 + \left(\frac{n_c}{n_g}\right)^2 \exp(-R_c^2 q^2). \tag{28}$$

The effective interferometry radius squared, corresponding to $C(q_{eff}) = 1 + 1/e$, is

$$R_{eff}^2 \equiv q_{eff}^{-2} = R_c^2/\left[1 + 2\ln(n_c(p)/n_g(p))\right]. \tag{29}$$

Finally, we can see that when the density in phase space increases the interferometry radius of the gas component reduces to $R/\sqrt{2}$ at most (see (26)). At very large densities, exceeding essentially the critical one, the Bose condensate determines the correlation function behavior at small p. Then the interferometry radius is reduced additionally as compared with its behavior for pure Bose-gas. At $\widetilde{n}/\widetilde{n}_{cr} \to \infty$, the interferometry radius in inclusive measurements tends to zero whatever large is the geometric size of the system. Note that intercept of the inclusive correlation function is equal to 2 and does not depend on $\widetilde{n}/\widetilde{n}_{cr}$. This reflects the chaotic nature of the thermal multiboson source. We are not discussing here the very complicated problem of spontaneous symmetry breaking of the condensate in finite systems which could take place when there is the interaction destroying the degeneration in the system.

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[1] W. Zajc, *Phys. Rev.* D **35,** 3396 (1987).
[2] W.N. Zhang *et al.*, *Phys. Rev.* C **47,** 795 (1993).
[3] S. Pratt, *Phys. Lett.* B **301,** 159 (1993); *Phys. Rev.* C **50,** 469 (1994).
[4] G.F. Bertsch, *Phys. Rev. Lett.* **72,** 2349 (1994).
[5] Yu.M. Sinyukov, B. Lorstad, *Z. Phys.* C **61,** 587 (1994).
[6] N.S. Amelin, R. Lednicky, *Heavy Ion Physics* **4,** 241 (1996); SUBATECH 95-08, Nantes 1995.
[7] B. Erazmus et al., Internal Note ALICE 95-43, Geneva 1995.
[8] W.Q. Chao, C.S. Gao, Q.H. Zhang, *J. Phys. G: Nucl. Phys.* **21,** 847 (1995).
[9] J. Zimanyi, T. Csorgo, hep-ph/9705432; T. Csorgo, J. Zimanyi, hep-ph/9705433.
[10] D.N.Zubarev, Nonequilibrium Statistical Thermodynamics, Nauka, Moscow, 1971.
[11] Ch.G.von Weert, *Ann.Phys.* **140,** 133 (1982).
[12] M.Gaudin, *Nucl.Phys.* **15,** 89 (1960).
[13] Yu.M.Sinyukov. Preprint ITP-93-8E (unpublished); *Nucl.Phys.* A **566,** 589c (1994); in: Hot Hadronic Matter:Theory and Experiment , (J.Letessier, H.H.Gutbrod, J.Rafelski, eds.) p. 309, Plenum Publ. Corp., 1995.